PMM U.S.S.R., Vol.47, No.6, pp.809-817, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 624.07

ON THE THEORY OF CURVILINEAR TIMOSHENKO-TYPE RODS*

V.L. BERDICHEVSKII and L.A. STAROSEL'SKII

An asymptotically exact theory of isotropic Timoshenko-type rods is developed. The variational problem is formulated for the cross-section in order to calculate the transverse shear coefficients. Shear coefficients are obtained for a series of transverse cross-sections.

1. One-dimensional theory of rods. In classical theory a rod is modelled by a curve Γ with a set of orthogonal reference vectors attached at each point, with one vector of each set tangential to Γ. The theory allows for four functionally independent degrees of freedom: three components $r^i(\xi)$ of the radius vector of the points on Γ (the small latin letters i, j, k take the values 1,2,3 and correspond to the projections on the Cartesian axes of the observer coordinate system, and ξ is a parameter on Γ) and six components of the reference vectors $\tau_\alpha^{\ i}$ orthogonal to Γ (the small greek letters take the values 1,2) connected by the following five relations:

$$\tau_\alpha^{\ i}\tau_{i\beta} = \delta_{\alpha\beta}, \quad \tau_\alpha^{\ i} r_{i,\xi} = 0 \tag{1.1}$$

where $\delta_{\alpha\beta}$ are the Kronecker deltas and the comma preceding ξ in the subscripts denotes differentiation with respect to ξ. The degree of freedom which exists when the set of reference vectors is defined, describes the relative rotation of the transverse cross-section. The curvatures ω_α and torsion ω of the rod are given by the relations

$$\tau^i_{,s} = -\omega^\alpha\tau_\alpha^{\ i}, \quad \tau^i_{\alpha,s} = \omega_\alpha\tau^i + \omega e_{\alpha}^{\cdot\beta}\tau_\beta^{\ i}$$

where τ^i denotes the unit vector tangent to Γ, s is the arc length along Γ, and the comma preceding s in the subscripts denotes differentiation with respect to s. The following quantities can be taken as the measures of the elongation, bending and torsion:

$$\gamma = \frac{1}{2}(s^2_{,\xi} - 1), \quad \Omega_\alpha = (1+2\gamma)^{1/2}\omega_\alpha - \omega_\alpha^\circ$$
$$\Omega = (1+2\gamma)^{1/2}\omega - \omega^\circ$$

The superscript $^\circ$ denotes quantities in the undeformed state, and we use the arc length on the curve Γ_0 as the parameter ξ. The formulas for γ, Ω_α and Ω are written in terms of $\tau_\alpha^{\ i}(\xi)$ and $r^i(\xi)$ as follows:

$$\gamma = \frac{1}{2}(r^i_{,\xi} r_{i,\xi} - 1), \quad \Omega_\alpha = r^i_{,\xi}\tau_{i\alpha,\xi} - \omega_\alpha^\circ, \quad \Omega = \frac{1}{2}e^{\alpha\beta}\tau^i_{\alpha,\xi}\tau_{i\beta} - \omega^\circ \tag{1.2}$$

The variational equation of the one-dimensional theory of rods has the form

$$\delta\int_{t_0}^{t_1}\int_0^{|\Gamma_0|}(K-\Phi)\,d\xi\,dt + \delta\int_{t_0}^{t_1}\int_0^{|\Gamma_0|}A\,d\xi\,dt = 0 \tag{1.3}$$

where $|\Gamma_0|$ is the rod arc length in the undeformed state, K and Φ are the kinetic and internal energy density per unit length, A is the work done by external forces, and r^i and $\tau_\alpha^{\ i}$, are the functions varied and obeying the relations (1.1).

In the classical theory of isotropic inhomogeneous rods (with centrally symmetric cross-section and even elastic properties) we have

$$2\Phi = \langle E\rangle\gamma^2 + \langle E\xi^\alpha\xi^\beta\rangle\Omega_\alpha\Omega_\beta + C\Omega^2 \tag{1.4}$$
$$2K = \langle\rho\rangle r_{,t}^{\ i} r_{i,t} + \langle\rho\xi^\alpha\xi^\beta\rangle\tau^i_{\alpha,t}\tau_{i\beta,t}$$

Here $\langle\cdot\rangle$ denotes the integral over the transverse cross-section, ξ^α are the Cartesian coordinates in the transverse cross-section, E is Young's modulus and ρ is the density of the material. To compute the torsional rigidity C we must solve the Saint-Venant problem at the cross-section (μ is the shear modulus and the comma preceding the greek subscripts denotes differentiation with respect to ξ^α)

*Prikl.Matem.Mekhan., 47, 6, 1015-1024, 1983

$$C\Omega^2 = \inf_g \langle \mu (g_{,\alpha} + \Omega e_{\gamma\alpha}\xi^\gamma)(g_,{}^\alpha + \Omega e_{\sigma}^{\cdot\alpha}\xi^\sigma)\rangle \tag{1.5}$$

Here $e_{\gamma\alpha}$ are the Levi-Civita symbols $(e_{11} = e_{22} = 0,\ e_{12} = -e_{21} = 1)$.

Formulas (1.4) were obtained for a homogeneous rod from the Kirchhoff hypothesis on plane flows /1/. They can be obtained by an asymptotic analysis of a three-dimensional energy functional, retaining in the expression for the energy the principal terms only and neglecting corrections of the order of $h/R, h/l$ and ε compared with unity /2/ where h is the diameter of transverse cross-section, R is the characteristic radius of torsional curvature, l is the characteristic scale of change in the stress state along the rod and ε is the deformation amplitude (the quantities R, l, ε are defined in /2/). Below we construct a more accurate theory, in which the expression for the energy retains corrections of the order of h/R, h/l and $(h/l)^2$ compared with unity.

In the improved theory the vectors $\tau_\alpha{}^i$ are not assumed to be orthogonal to the vector τ^i and two additional degrees of freedom are introduced, namely the transverse shear $\varphi_\alpha = \tau^i\tau_{i\alpha}$ (this idea is due to S.P. Timoshenko /3/). The vectors $\tau_\alpha{}^i$ are of unit length and mutally orthogonal $(\tau_\alpha{}^i\tau_{i\beta} = \delta_{\alpha\beta})$ just as in the classical theory. The measures of flexure and torsion are found in terms of the vectors $r^i(\xi), \tau_\alpha{}^i(\xi)$ by means of the formulas (1.2). The internal energy density contains, in the improved theory, apart from the classical terms, the cross terms connecting the elongation with torsion, elongation with flexure, flexure with torsion and shear energy

$$2\Phi = \langle E\rangle \gamma^2 + \langle E\xi^\alpha\xi^\beta\rangle \Omega_\alpha\Omega_\beta + C\Omega^2 + 2\,[\langle E\xi^\alpha\xi_\alpha\rangle - \tag{1.6}$$

$$2\,(1+\nu)\,C]\ \omega^\circ\gamma\Omega - 2\,\langle E\xi^\alpha\xi^\beta\rangle \sqrt{C_1/C}\ \omega_\gamma{}^\circ e_{\alpha\cdot}{}^{\gamma}\Omega_\beta\Omega - 2D^{\alpha\beta}\omega_\alpha{}^\circ\gamma\Omega_\beta + J^{\alpha\beta}\varphi_\alpha\varphi_\beta$$

The cross term connecting the elongation with torsion exists only for naturally twisted rods $(\omega^\circ \neq 0)$ and was first computed in /4/, while in /5/ it was obtained from the asymptotic representations. The quantity $\langle E\xi^\alpha\xi_\alpha\rangle - 2\,(1+\nu)\,C$ characterizing the interaction between the elongation and torsion is computed from the moduli of rigidity of the classical theory $\langle E\xi^\alpha\xi^\beta\rangle$ and C. In contrast, the effect of transverse shear is connected with three conditional independent characteristics of the rod, namely with the shear rigidities $J^{\alpha\beta}$. Below we show that the latter are obtained from the solution of the following variational problem at the cross-section:

$$J^{\alpha\beta}\varphi_\alpha\varphi_\beta = \inf_z \langle \mu\ (\varphi_\alpha + z_{,\alpha})\ (\varphi^\alpha + z_,{}^\alpha)\rangle \tag{1.7}$$

The minimum in (1.7) is sought over all functions $z(\xi^\alpha)$ satisfying the restriction $\langle \mu z\xi^\alpha\rangle = 0$, φ_α are regarded as constant parameters. The transverse shear coefficients $K^{\alpha\beta} = J^{\alpha\beta}/\langle\mu\rangle$ are, as a rule, of the order of unity, and the shear energy $J^{\alpha\beta}\varphi_\alpha\varphi_\beta$ represents formally the principal part of the energy. Therefore, in the first step of the variational-asymptotic method /6/ we must minimize the shear energy in accordance with the hypothesis of plane flows $\varphi_\alpha = 0$ also in the first approximation. However, cases are possible (one such case is discussed below) when $K^{\alpha\beta} \ll 1$, and the theory derived below, regarded as improved, becomes a first approximation theory.

The rods which are curvilinear in the undeformed state are characterized by another five parameters, i.e. four components of the non-symmetric tensor $D^{\alpha\beta}$ and scalar C_1, connected with the cross relation between the elongation and flexure, and flexure and torsion. To find $D^{\alpha\beta}$ we must obtain the coefficients of the quadratic form $J^{\alpha\beta}\varphi_\alpha\varphi_\beta + B^{\alpha\beta}\varphi_\alpha\theta_\beta + C^{\alpha\beta}\theta_\alpha\theta_\beta$ representing the minimum value of the functional

$$\left\langle \mu \left(z_{,\alpha} + \varphi_\alpha - \frac{\nu}{2}\chi_\alpha{}^\nu\theta_\nu\right)\left(z_,{}^\alpha + \varphi^\alpha - \frac{\nu}{2}\chi^{\alpha\sigma}\theta_\sigma\right)\right\rangle \tag{1.8}$$

$$\chi_\alpha{}^\beta = 2\,(\xi^\beta\xi_\alpha - \langle\mu\xi^\beta\xi_\alpha\rangle/\langle\mu\rangle) - (\xi_\nu\xi^\nu - \langle\mu\xi_\nu\xi^\nu\rangle/\langle\mu\rangle)\,\delta_\alpha{}^\beta$$

Here φ_α and θ_α are constant parameters and $\chi_\alpha{}^\beta$ are quadratic polynomials. The minimum is sought over all z satisfying the restriction $\langle\mu z\xi^\alpha\rangle = 0$. The tensor $D^{\alpha\beta}$ is determined in terms of the coefficients of the quadratic form $J^{\alpha\beta}, B^{\alpha\beta}, C^{\alpha\beta}$ from the formulas

$$D^{\alpha\beta} = (2+\nu)\,\langle E\xi^\alpha\xi^\beta\rangle + \langle E\rangle H^{\alpha\beta},\quad H_{\alpha\cdot}^{\cdot\beta} = {}^1/_2 J_{\alpha\gamma}^{(-1)}B^{\gamma\beta} + \lambda_\alpha{}^\beta \tag{1.9}$$

$$\lambda^{11} = {}^1/_2\,\langle E\xi_1\xi_1\rangle^{-1}\,(C^{11} - {}^1/_4 J_{\alpha\beta}^{(-1)}B^{\alpha 1}B^{\beta 1}),\quad \lambda^{22} = {}^1/_2\,\langle E\xi_2\xi_2\rangle^{-1}$$

$$(C^{22} - {}^1/_4 J_{\alpha\beta}^{(-1)}B^{\alpha 2}B^{\beta 2}),\quad \lambda^{12} = \lambda^{21} = \langle E\xi_\alpha\xi^\alpha\rangle^{-1}\,(C^{12} - {}^1/_4 J_{\alpha\beta}^{(-1)}B^{\alpha 1}B^{\beta 2})$$

where $J_{\alpha\beta}^{(-1)}$ is a tensor inverse to $J^{\alpha\beta}(J_{\alpha\beta}^{(-1)}J^{\beta\gamma} = \delta_\alpha{}^\gamma)$. To find the scalar C_1, we must solve the problem of the minimum of the functional

$$\langle\lambda\ (z_{,\alpha}{}^\alpha + 2\nu g)^2 + 2\mu\ (z_{(\alpha,\beta)} + \nu\delta_{\alpha\beta}g)\ (z^{(\alpha,\beta)} + \nu\delta^{\alpha\beta}g) - \tag{1.10}$$

$$2\mu\ (g_{,\alpha} + e_{\gamma\alpha}\xi^\gamma)\ z^\alpha\rangle$$

where g is the smallest element in the problem of torsion (1.5). The minimum is sought over all z^α satisfying the restriction (3.3). Let us denote the minimum value of the functional (1.10) by J. Then C_1 is given by the formula

$$C_1 = \langle Eg^2\rangle + J \tag{1.11}$$

Just as in the case of shells /7, 8/, the cross term connecting the elongation with flexure can be essential in determining the displacements to a first approximation. Formulas (1.6)–(1.11) are obtained assuming that only the shear modulus changes in transverse directions and Poisson's ratio is constant. Note that the asymptotic theory of rods, with transverse shear taken into account, was also constructed in /9/.

2. The three-dimensional problem. We consider, in the Cartesian x^i coordinate system, an elastic isotropic rod occupying in its undeformed state a volume V_0 formed by the motion along the rod axis Γ_0 of a plane figure S perpendicular to Γ_0 at every point of the axis. The centre of gravity of S lies on the rod axis. The rod is acted upon by the time-dependent surface forces P_i and mass forces F_i which are assumed to be dead. Let us introduce in the volume V_0 an associated coordinate system ξ^α, $\xi^3 = \xi$ according to the formulas

$$x^{0i}(\xi, \xi^\alpha) = r^{0i}(\xi) + \tau_\alpha^{0i}(\xi)\,\xi^\alpha \tag{2.1}$$

Here $x^{0i} = r^{0i}(\xi)$ is the equation of the rod axis, Γ_0, ξ is the arc length along the axis, τ_1^{0i} and τ_2^{0i} are the components of two vectors which form, together with $\tau^{0i} = r_{,\xi}^{0i}$ an orthogonal set of vectors. We assume that S is centrally symmetric. This means that in addition to every point with coordinates ξ^α it contains a point with coordinates ξ^α. The distribution of the inhomogeneities over the cross-section will aso be assumed symmetric (i.e. Young's modulus $E(\xi^\alpha)$ and shear modulus $\mu(\xi^\alpha)$ are even functions of the coordinates ξ^α) and Poisson's ratio ν will be assumed constant.

The equations describing the deformation of the rod follow from the variational equation

$$\delta\int_{t_0}^{t_1}(I - L)\,dt = 0, \quad I = \int_0^{|\Gamma_0|}\langle\Lambda\sqrt{g^\circ}\rangle\,d\xi \tag{2.2}$$

$$L = \int_0^{|\Gamma_0|}(\langle F_i x^i\sqrt{g^\circ}\rangle + \langle P_i x^i\rangle_{\partial S})\,d\xi + \langle P_i x^i\rangle\Big|_{\xi=0}^{\xi=|\Gamma_0|}$$

$$\Lambda = \frac{1}{2}[\lambda(g^{\circ ab}\varepsilon_{ab})^2 + 2\mu g^{\circ ab}g^{\circ cd}\varepsilon_{ac}\varepsilon_{bd}] - \frac{1}{2}\rho v^i v_i, \quad g^\circ = \det\|g^{\circ ab}\|$$

Here $\langle\cdot\rangle_{\partial S}$ denotes the integral over the boundary of S, $x^i(\xi^\alpha, t)$ is the law of motion of the body, $v^i = x_{,t}^i$, Λ is the difference between the internal and kinetic energy, L is a functional determining the work done by external mass and surface forces, and g_{ab}°, $g^{\circ ab}$ are components of the metric tensor. The components of the strain tensor are determined from the law of motion using the formula

$$\varepsilon_{ab} = \frac{1}{2}(x_{,a}^i x_{i,b} - \overset{\circ}{g}_{ab}) \tag{2.3}$$

The problem in question reduces to replacing the three-dimensional problem of the theory of elasticity by the appropriate "one-dimensional" problem containing the functions of the longitudinal coordinate ξ and time t only. The one-dimensional theory can be regarded as a result of the passage to the limit $h \to 0$. We will construct the one-dimensional theory using the variational-asymptotic method /6/.

3. Asymptotic analysis of the three-dimensional problem. *Transformation of the expression for the energy.* We will write U in the form of a sum of three positive definite quadratic forms, i.e. the longitudinal energy $U_\parallel$, the transverse energy $U_\perp$ and the shear energy $U_<$

$$U = U_\parallel + U_\perp + U_<$$
$$U_\parallel = {}^1/_2 E_\parallel \varepsilon_{33}^2, \quad U_\perp = {}^1/_2 E^{\alpha\beta\gamma\delta}(\varepsilon_{\alpha\beta} + E_{\alpha\beta}\varepsilon_{33} + E_{\alpha\beta}^\sigma\varepsilon_{\sigma 3})(\alpha, \beta \to \gamma, \delta)$$
$$U_< = {}^1/_2 G^{\alpha\beta}(\varepsilon_{\alpha 3} + E_\alpha\varepsilon_{33})(\alpha \to \beta)$$

The symbol $(\alpha, \beta \to \gamma, \delta)$ describes the expression within the preceding brackets in which the subscripts α, β have been replaced by γ, δ. The components of the "two-dimensional" tensors of elastic moduli are expressed in terms of the metric tensor components, Lamé paraemters λ, μ and Poisson's ratio ν

$$E_\parallel = 4\mu\frac{g^{\circ 33}}{g_{33}^\circ} + 2\mu(\nu - 1)\frac{1}{g_{33}^{\circ 2}} - 4\mu\frac{e_{\alpha\nu}e_{\sigma\rho}g^{\circ\nu\rho}g^{\circ 3\alpha}g^{\circ 3\sigma}}{g_{33}^{\circ 2}e_{\tau\beta}e_{\gamma\delta}\left[g^{\circ 3\tau}g^{\circ 3\gamma} + \frac{1}{2}(2/g_{33}^\circ - g^{\circ 33})g^{\circ\tau\gamma}\right]g^{\circ\beta\delta}}$$

$$G^{\alpha\beta} = 4\mu\,[g^{\circ 3\alpha} g^{\circ 3\beta} + (2/g_{33}{}^{\circ} - g^{\circ 33})\, g^{\circ\alpha\beta}],\quad E^{\sigma}_{\alpha\beta} = -\frac{2}{g_{33}{}^{\circ}}\, \overset{\circ}{g}_{3(\alpha}\delta^{\sigma}_{\beta)}$$

$$E_{\alpha} = \frac{e_{\alpha\nu} e_{\sigma\rho} g^{\circ\nu\rho} g^{\circ 3\sigma}}{g_{33}{}^{\circ} e_{\tau\beta} e_{\gamma\delta} \left[g^{\circ 3\tau} g^{\circ 3\gamma} + \frac{1}{2}(2/g_{33}{}^{\circ} - g^{\circ 33})\, g^{\circ\tau\gamma} \right] g^{\circ\beta\delta}}$$

$$E^{\alpha\beta\gamma\delta} = \lambda g^{\circ\alpha\beta} g^{\circ\gamma\delta} + \mu\,(g^{\circ\alpha\gamma} g^{\circ\beta\delta} + g^{\circ\beta\gamma} g^{\circ\alpha\delta})$$

$$E_{\alpha\beta} = \frac{\nu}{g_{33}{}^{\circ}}\, \overset{\circ}{g}_{\alpha\beta} + \frac{(1-\nu)}{g_{33}{}^{\circ 2}}\, \overset{\circ}{g}_{\alpha 3} \overset{\circ}{g}_{\beta 3}$$

Remembering that the metric tensor components are given in the ξ^{α} coordinate system by the formulas

$$g_{33}{}^{\circ} = (1 + \omega_{\alpha}{}^{\circ}\xi^{\alpha})^2 + \omega^{\circ 2}\xi_{\alpha}\xi^{\alpha},\quad g^{\circ 33} = (1 + \omega_{\alpha}{}^{\circ}\xi^{\alpha})^{-2}$$

$$\overset{\circ}{g}_{\alpha 3} = \omega^{\circ} e_{\beta\alpha}\xi^{\beta}$$

$$g^{\circ\alpha 3} = \omega^{\circ} e_{\beta}{}^{\alpha}\xi^{\beta}(1 + \omega_{\gamma}{}^{\circ}\xi^{\gamma})^{-2},\quad \overset{\circ}{g}_{\alpha\beta} = \delta_{\alpha\beta}$$

$$g^{\circ\alpha\beta} = \delta_{\alpha\beta} + \omega^{\circ 2}(\delta_{\alpha\beta}\xi_{\gamma}\xi^{\gamma} - \xi_{\alpha}\xi_{\beta})(1 + \omega_{\nu}{}^{\circ}\xi^{\nu})^{-2}$$

$$g^{\circ} = \det \| \overset{\circ}{g}_{ab} \| = (1 + \omega_{\alpha}{}^{\circ}\xi^{\alpha})^{-2}$$

and neglecting terms of the order of $(h/R)^2$ compared with unity, we obtain the following expressions for the components of the two-dimensional elastic moduli:

$$E_{\|} = E\,(1 - 4\omega_{\alpha}{}^{\circ}\xi^{\alpha}),\quad E^{\alpha\beta\gamma\delta} = E_0^{\alpha\beta\gamma\delta} = \lambda\delta^{\alpha\beta}\delta^{\gamma\delta} + \mu\,(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\beta\gamma}\delta^{\alpha\delta})$$

$$E_{\alpha\beta} = \nu\delta_{\alpha\beta}(1 - 2\omega_{\gamma}{}^{\circ}\xi^{\gamma}),\quad G^{\alpha\beta} = 4\mu\delta^{\alpha\beta}(1 - 2\omega_{\gamma}{}^{\circ}\xi^{\gamma})$$

$$E^{\sigma}_{\alpha\beta} = 2\omega^{\circ} e_{(\alpha\tau}\delta^{\sigma}_{\beta)}\xi^{\tau},\quad E_{\alpha} = \omega^{\circ} e_{\alpha\beta}\xi^{\beta}$$

External forces.
We will assume that the external surface and mass forces are of the order of

$$P_i = O\left(\frac{h}{l}\mu\varepsilon\right),\quad F_i = O\left(\frac{1}{l}\mu\varepsilon\right)$$

and the mass forces F_i are constant over the cross-section S.

Let us assume that the relation $P_i(\xi^{\alpha})$ at the end faces can be written in the form $P_i = \mu\,(C_i + C_{i\alpha}\xi^{\alpha})$ where $C_i, C_{i\alpha} = \text{const}$. We begin the asymptotic analysis of the three-dimensional problem by considering the static case.

First approximation. As was shown in /10/, the law of motion to a first approximation has the form

$$x^i(\xi^{\alpha}, \xi) = r^i(\xi) + h\tau_{\alpha}{}^i\zeta^{\alpha} + hy^i,\quad \zeta^{\alpha} = h^{-1}\xi^{\alpha} \tag{3.1}$$

$$y_{\alpha} = \tau_{\alpha}{}^i y_i = -\nu\left(\zeta_{\alpha}\gamma + \frac{1}{2}\chi_{\alpha}{}^{\beta} h\Omega_{\beta}\right),\quad y = \tau^i \quad y_i = gh\Omega$$

where g is the minimizing element in the variational problem (1.5). Substituting (3.1) into the expression for the energy and integrating over the transverse cross-section, we obtain the formula for Φ (1.4). Subsequent terms of the expansion (3.1) are of the order of $\varepsilon h/l$ and made only a small contribution to the energy.

Second approximation. In accordance with the general scheme of the variational-asymptotic method we write the law of motion in the form

$$x^i(\xi^{\alpha}, \xi) = r^i + h\tau_{\alpha}{}^i\zeta^{\alpha} + hy^i + hz^i,\quad y^i = \tau^i y + \tau_{\alpha}{}^i y^{\alpha} \tag{3.2}$$

The arbitrariness in the choice of r^i and $\tau_{\alpha}{}^i$ makes it possible to impose the following restrictions on z^i:

$$\langle \mu z^i \rangle = 0,\ \langle \mu z_{\alpha|\beta} \rangle e^{\alpha\beta} = 0,\ \langle \mu z\zeta^{\alpha} \rangle = 0\ (z_{\alpha} = \tau_{\alpha}{}^i z_i,\ z = \tau^i z_i) \tag{3.3}$$

where the symbol $|\beta$ denotes differentiation with respect to ζ^{β}. Let us substitute (3.2) into the expression for the strain tensor components (2.3). Neglecting quantities of the order of h^2/Rl and ε compared with unity, we obtain

$$\varepsilon_{33} = {}^1/_2(2\gamma + 2h\Omega_{\alpha}\zeta^{\alpha} + 2h^2\omega_{\alpha}{}^{\circ}\Omega_{\beta}\zeta^{\alpha}\zeta^{\beta} + 2h^2\omega^{\circ}\Omega\zeta_{\alpha}\zeta^{\alpha} + 2h\tau^i y_{i,\xi} + 2h\tau^i z_{i,\xi}) \tag{3.4}$$

$$\varepsilon_{\alpha 3} = {}^1/_2[h\Omega(g_{|\alpha} + \zeta^{\gamma} e_{\gamma\alpha}) + \varphi_{\alpha} + h\zeta^{\gamma}\tau^i_{\gamma,\xi} y_{i|\alpha} + z_{|\alpha} + h\zeta^{\gamma}\tau^i_{\gamma,\xi} z_{i|\alpha} + hy^i_{,\xi}\tau_{i\alpha} + h\tau_{\alpha}{}^i z_{i,\xi}],\quad \varepsilon_{\alpha\beta} = y_{(\alpha|\beta)} + z_{(\alpha|\beta)}$$

Let us separate, from the energy functional, the terms containing z and z_α , principal in the asymptotic sense. By (3.4) the terms have the form

$$\langle {}^1/_2 E_0^{\alpha\beta\gamma\delta}(z_{\alpha|\beta} + \nu\delta_{\alpha\beta} g h^2\Omega_{,\xi})(\alpha, \beta \to \gamma, \delta) - \mu h^2\Omega_{,\xi}(g_{|\alpha} + \zeta^\gamma e_{\gamma\alpha}) z^\alpha + {}^1/_2 \mu\delta^{\alpha\beta}(z_{|\alpha} + \varphi_\alpha + h y_{\alpha,\xi})(\alpha \to \beta)\rangle - \langle P_i(\tau^i z + \tau_\alpha{}^i z^\alpha)\rangle_{\partial S} \tag{3.5}$$

Let us carry out the following substitution of the function sought:

$$z = z' + \frac{1}{2}\nu(\zeta_\alpha\zeta^\alpha - \langle\mu\zeta_\alpha\zeta^\alpha\rangle/\langle\mu\rangle) h\gamma_{,\xi} - \varphi_\alpha\zeta^\alpha$$

In terms of the functions z', z_α the principal terms take the form

$$\langle {}^1/_2 E_0^{\alpha\beta\gamma\delta}(z_{\alpha|\beta} + \nu\delta_{\alpha\beta} g h^2\Omega_{,\xi})(\alpha, \beta \to \gamma, \delta) - \mu h^2\Omega_{,\xi}(g_{|\alpha} + \zeta^\gamma e_{\gamma\alpha}) z^\alpha + {}^1/_2 \mu\delta^{\alpha\beta}\left(z'_{|\alpha} - \frac{\nu}{2}\chi_\alpha{}^\gamma h^2\Omega_{\gamma,\xi}\right)(\alpha \to \beta)\rangle - \langle P_i(\tau^i z' + \tau_\alpha{}^i z^\alpha)\rangle_{\partial S} \tag{3.6}$$

The restrictions (3.3) for the function z' become

$$\langle\mu z'\rangle = 0,\ \langle\mu z'\zeta^\beta\rangle - \varphi_\alpha\langle\mu\zeta^\alpha\zeta^\beta\rangle = 0 \tag{3.7}$$

Minimizing the functional (3.6) with restrictions (3.7), we obtain the variational problems (1.7), (1.8), (1.10). A minimum is attained on the functions

$$z = \frac{1}{2}\nu(\zeta_\alpha\zeta^\alpha - \langle\mu\zeta_\alpha\zeta^\alpha\rangle/\langle\mu\rangle) h\gamma_{,\xi} - \varphi_\alpha(\zeta^\alpha - A_{\tau\cdot}^{\cdot\alpha} g^\tau) + \nu(e^\gamma - S_{\tau\cdot}^{\cdot\gamma} g^\tau) h^2\Omega_{\gamma,\xi} + f - \varkappa_{\alpha\beta}\langle f\zeta^\beta\rangle g^\alpha, \quad z_\alpha = z_\alpha{}^\circ h^2\Omega_{,\xi} + f_\alpha \tag{3.8}$$

where the functions $g^\tau, e^\gamma, f, z_\alpha{}^\circ, f_\alpha$ represent the solutions of the Euler equations corresponding to the variational problem (3.6), and $A_{\tau\cdot}^{\cdot\alpha}$ and $S_{\tau\cdot}^{\cdot\gamma}$ are constant tensors given by the formulas

$$A_{\tau\cdot}^{\cdot\alpha} = \varkappa_{\tau\beta}\langle\mu\zeta^\beta\zeta^\alpha\rangle, \quad S_{\tau\cdot}^{\cdot\gamma} = \varkappa_{\tau\beta}\langle\mu e^\gamma\zeta^\beta\rangle, \quad \varkappa_{\tau\beta} = \langle\mu\zeta^\beta g^\tau\rangle^{-1}$$

Thus at the second step of the variational-asymptotic method the law of motion is determined up to and including terms of the order of $\varepsilon h/l$. It can be shown that this is sufficient to construct a theory including in the expression for the energy corrections of the order of h/R and $(h/l)^2$ compared with unity.

Substituting (3.2) into the variational equation (2.2) and retaining terms of the necessary order, we obtain the variational equation (1.3) with the densities of internal energy and work done by the forces of the form

$$\begin{aligned}
\Phi &= \frac{1}{2}\langle E\rangle\gamma^2 + \frac{1}{2}\langle E\xi^\alpha\xi^\beta\rangle\Omega_\alpha\Omega_\beta + \frac{1}{2}C\Omega^2 - (2+\nu)\langle E\xi^\alpha\xi^\beta\rangle\omega_\alpha{}^\circ\Omega_\beta\gamma + [\langle E\xi^\alpha\xi_\alpha\rangle - 2(1+\nu)C]\omega^\circ\gamma\Omega + \\
&\quad \frac{1}{2}C_1\Omega_{,\xi}^2 + \frac{1}{2}(J^{\alpha\beta}\varphi_\alpha\varphi_\beta + B^{\alpha\beta}\varphi_\alpha\Omega_{\beta,\xi} + C^{\alpha\beta}\Omega_{\alpha,\xi}\Omega_{\beta,\xi}) \\
A &= Q^i r_i + (Q_i{}^\alpha + \omega_\beta{}^\circ I^{\alpha\beta}F_i)\tau_\alpha{}^i + R\Omega + R^\alpha\Omega_\alpha + N\gamma + N^\alpha\varphi_\alpha + (T_i r^i + T_i{}^\alpha\tau_\alpha{}^i)\Big|_{\xi=0}^{\xi=|\Gamma_\circ|} \\
R &= \langle P_i g\rangle_{\partial S}\tau^i - \langle P_{i,\xi} z^{\circ\alpha}\rangle\tau_\alpha{}^i - a e^{\alpha\beta}\langle\mu f_{\alpha,\beta\xi}\rangle \\
a &= e_{\alpha\beta}\langle\mu(g_{,}{}^\alpha + e_{\rho\cdot}^{\cdot\alpha}\xi^\rho)\xi^\beta\rangle/(2\langle\mu\rangle), \\
R^\alpha &= -(\nu/2)\langle P_i\chi^{\alpha\beta}\rangle_{\partial S}\tau_\beta{}^i - \nu\langle P_{i,\xi}(e^\alpha - S_{\beta\cdot}^{\cdot\alpha} g^\beta)\rangle_{\partial S}\tau^i \\
N &= -\nu\left[Q_i{}^\alpha\tau_\alpha{}^i + \frac{1}{2}\langle P_{i,\xi}(\xi^\alpha\xi_\alpha - \langle\mu\xi_\alpha\xi^\alpha\rangle/\langle\mu\rangle)\rangle_{\partial S}\tau^i\right] \\
N^\alpha &= -Q_i{}^\alpha\tau^i + A_{\beta\cdot}^{\cdot\alpha}\langle P_i g^\beta\rangle_{\partial S}\tau^i \\
Q^i &= |S|F^i + \langle P^i\rangle_{\partial S}, \quad Q_i{}^\alpha = \langle P_i\xi^\alpha\rangle_{\partial S}, \quad T_i = \langle P_i\rangle \\
T_i{}^\alpha &= \langle P_i\xi^\alpha\rangle
\end{aligned} \tag{3.9}$$

In the homogeneous case the formula for Φ (when $\gamma = 0$) was obtained in /11/, and in the framework of the linear theory of rectilinear rods, in fact, in /12/. However, neither in /12/ nor in /11/ was the possibility mentioned of the transformation carried out below, and expression (3.9) was not reduced to its final simple form (1.6).

Transformation of the variational equation. We will simplify the expression for the energy by carrying out a substitution of the functions sought. We will redefine the transverse shear

$$\varphi_\alpha = \bar\varphi_\alpha + \lambda_{\alpha\cdot}^{\cdot\gamma}\Omega_{\gamma,\xi}$$

where $\lambda_{\alpha\cdot}^{\cdot\gamma}$ is a constant tensor, selected, in what follows, in a special manner. Then the flexural measure can be rewritten thus

$$\Omega_\alpha = \bar{\Omega}_\alpha + \lambda_{\alpha}^{\cdot\gamma}\Omega_{\gamma,\xi\xi}, \quad \bar{\Omega}_\alpha = \bar{\varphi}_{\alpha,\xi} - \tau^i_{,\xi}\tau_{i\alpha} - \omega_\alpha{}^\circ \tag{3.10}$$

Taking into account (3.10) we can write the group of terms from the expression for the energy density Φ (3.9) in the form

$$\frac{1}{2}\langle E\xi^\alpha\xi^\beta\rangle\Omega_\alpha\Omega_\beta + \frac{1}{2}(J^{\alpha\beta}\varphi_\alpha\varphi_\beta + B^{\alpha\beta}\varphi_\alpha\Omega_{\beta,\xi} + C^{\alpha\beta}\Omega_{\alpha,\xi}\Omega_{\beta,\xi}) = \tag{3.11}$$
$$\frac{1}{2}\langle E\xi^\alpha\xi^\beta\rangle\bar{\Omega}_\alpha\bar{\Omega}_\beta + \frac{1}{2}(J^{\alpha\beta}\bar{\varphi}_\alpha\bar{\varphi}_\beta + 2\Gamma^{\alpha\beta}\bar{\varphi}_\alpha\Omega_{\beta,\xi} + E^{\alpha\beta}\Omega_{\alpha,\xi}\Omega_{\beta,\xi})$$

$$\Gamma^{\alpha\beta} = \frac{1}{2}B^{\alpha\beta} + J^{\alpha\gamma}\lambda_{\gamma\cdot}^{\cdot\beta}, \quad E^{\alpha\beta} = C^{\alpha\beta} + B^{\gamma(\beta}\lambda_{\gamma\cdot}^{\cdot\alpha)} + J^{\gamma\delta}\lambda_{\gamma\cdot}^{\cdot\alpha}\lambda_{\delta\cdot}^{\cdot\beta} - 2\langle E\xi^{(\alpha}\xi^{\gamma\rangle}\lambda_{\gamma\cdot}^{\cdot\beta)}$$

where we omit the divergent term $(\langle E\xi^\alpha\xi^\beta\rangle\lambda_\beta{}^\delta\Omega_\alpha\Omega_{\delta,\xi})_{,\xi}$. Let us write the sum on the right-hand side of (3.11) as a quadratic form in $\bar{\varphi}_\alpha + H_{\alpha}^{\cdot\gamma}\Omega_{\gamma,\xi}$:

$$J^{\alpha\beta}\bar{\varphi}_\alpha\bar{\varphi}_\beta + 2\Gamma^{\alpha\beta}\bar{\varphi}_\alpha\Omega_{\beta,\xi} + E^{\alpha\beta}\Omega_{\alpha,\xi}\Omega_{\beta,\xi} = J^{\alpha\beta}(\bar{\varphi}_\alpha + H_{\alpha\cdot}^{\cdot\gamma}\Omega_{\gamma,\xi})\ (\alpha \to \beta)$$

The tensors $\Gamma^{\alpha\beta}$, $E^{\alpha\beta}$ and $H_\alpha{}^{\cdot\beta}$ are connected by the equations

$$\Gamma^{\alpha\beta} = J^{\alpha\gamma}H_{\gamma\cdot}^{\cdot\beta}, \quad E^{\alpha\beta} = J^{\gamma\delta}H_{\gamma\cdot}^{\cdot\alpha}H_{\delta\cdot}^{\cdot\beta} \tag{3.12}$$

Solving the first relation of (3.12) for the tensor $H_\gamma{}^{\cdot\beta}$ and substituting the result in the same relation, we obtain

$$E^{\alpha\beta} = J^{(-1)}_{\nu\rho}\Gamma^{\nu\alpha}\Gamma^{\rho\beta} \tag{3.13}$$

We satisfy the relation (3.13) by an appropriate choice of the tensor $\lambda_\alpha{}^{\cdot\beta}$ which we shall assume to be symmetric. Substituting into (3.13) the values of the terms appearing in it as given by (3.11), we obtain

$$\langle E\xi^{(\alpha}\xi^{\gamma\rangle}\lambda_\gamma^{\beta)} = \frac{1}{2}\Big(C^{\alpha\beta} - \frac{1}{4}J^{(-1)}_{\nu\rho}B^{\nu\alpha}B^{\rho\beta}\Big) \tag{3.14}$$

Expression (3.14) represents a system of three linear equations for the components of the tensor $\lambda_\gamma{}^\beta$. It can be shown that the determinant of (3.14) is not zero. Let us suppose that the axes of the associated coordinate system ξ^α coincide with the axes of the cross-section S in which the tensor $\langle E\xi^\alpha\xi^\beta\rangle$ is diagonal. Then (3.14) separates into three independent equations which yield the values of $\lambda_\alpha{}^\beta$ (1.9). Let us carry out another substitution of the functions sought, namely $\bar{\varphi}_\alpha \to \bar{\bar{\varphi}}_\alpha$

$$\bar{\bar{\varphi}}_\alpha = \bar{\varphi}_\alpha + H_{\alpha\cdot}^{\cdot\gamma}\Omega_{\gamma,\xi}$$

Replacing in the flexural measure $\bar{\Omega}_\alpha$ (3.10) $\bar{\varphi}_\alpha$ by $\bar{\bar{\varphi}}_\alpha$, we obtain

$$\bar{\Omega}_\alpha = \bar{\bar{\varphi}}_{\alpha,\xi} - \tau^i_{,\xi}\tau_{i\alpha} - \omega_\alpha{}^\circ - H_{\alpha\cdot}^{\cdot\gamma}\Omega_{\gamma,\xi\xi} \tag{3.15}$$

The expression for the flexural measure will be the same as the earlier expression if we redefine simultaneously the components of the radius vector r^i: $r^i \to \bar{r}^i$, $r^i = \bar{r}^i - \tau_\nu{}^i H^{\nu\gamma}\Omega_\gamma$. We note that within the accuracy used we can also represent the substitution of the unknown functions in the form

$$r^i = \bar{r}^i - \tau_\nu{}^i H^{\nu\gamma}\bar{\Omega}_\gamma \tag{3.16}$$

We will now write the expression for the flexural measure with an accuracy of the order of $\varepsilon h^2/Rl$ and $(h/R)^2\varepsilon$ as follows:

$$\bar{\Omega}_\alpha = \bar{\bar{\varphi}}_{\alpha,\xi} - \bar{\tau}^i_{,\xi}\tau_{i\alpha} - \omega_\alpha{}^\circ = \bar{\tau}^i\tau_{i\alpha,\xi} - \omega_\alpha{}^\circ, \quad \bar{\tau}^i = \bar{r}^i_{,\xi}, \quad \bar{\tau}^i\tau_{i\alpha} = \bar{\bar{\varphi}}_\alpha$$

Let us write the group of terms in the energy density connected with the torsion, in the form

$$\frac{1}{2}C\Omega^2 + \frac{1}{2}C_1\Omega^2_{,\xi} = \frac{1}{2}C(\Omega + \sqrt{C_1/C}\,\Omega_{,\xi})^2$$

where we omit the divergent term $(\sqrt{CC_1}\Omega^2)_{,\xi}$, and make another substitution of the unknown functions $\tau_\alpha{}^i \to \bar{\tau}_\alpha{}^i$

$$\tau_\alpha{}^i = \bar{\tau}_\alpha{}^i - \bar{\tau}_\beta{}^i e_{\alpha\cdot}^{\cdot\beta}\sqrt{C_1/C}\,\Omega \tag{3.17}$$

with the vectors $\bar{\tau}^i, \bar{\tau}_\alpha{}^i$ satisfying the relations

$$\bar{\tau}^i\bar{\tau}_{i\alpha} = \bar{\bar{\varphi}}_\alpha + O(\varepsilon^2), \quad \bar{\tau}_\alpha{}^i\bar{\tau}_{i\beta} = \delta_{\alpha\beta} + O(\varepsilon)$$

Using these relations we can show that the following relation holds:

$$\Omega + \sqrt{C_1/C}\,\Omega_{,\xi} = \bar{\Omega} + O(\varepsilon^2), \quad \bar{\Omega} = \frac{1}{2}e^{\alpha\beta}\bar{\tau}^i_{\alpha,\xi}\bar{\tau}_{i\beta}$$

From (3.17) it follows that

$$\tau_\alpha{}^i = \bar{\tau}_\alpha{}^i - \bar{\tau}_\beta{}^i e_{\alpha\cdot}^{\cdot\beta} \sqrt{C_1/C}\,\bar{\Omega} + \bar{\tau}_\beta{}^i e_{\alpha\cdot}^{\cdot\beta} C_1/C \bar{\Omega}_{,\xi} + O((h/l)^2 \varepsilon) \tag{3.18}$$

Taking into account (3.16) and (3.18), we can write the expressions for the flexural measures $\bar{\Omega}_\alpha$ and elongation of the γ-axis in the form

$$\bar{\Omega}_\alpha = \bar{\bar{\Omega}}_\alpha - \omega_\gamma{}^\circ e_{\alpha\cdot}^{\cdot\gamma} \sqrt{C_1/C}\,\bar{\Omega}, \quad \bar{\bar{\Omega}}_\alpha = \bar{\tau}^i \tau_{i\alpha,\xi} - \omega_\alpha{}^\circ$$

$$\gamma = \bar{\gamma} - \omega_\gamma{}^\circ H^{\gamma\beta} \bar{\bar{\Omega}}_\beta$$

After these transformations and substitutions of the unknown functions, the energy density (3.9) is reduced (the bars are omitted) to the form (1.6), and the work done by external forces A (apart from the divergent terms) to the form (3.9) with different values of the effective forces R and R^α

$$R = \langle P_i g\rangle_{\partial S} \tau^i - Q_i{}^\alpha \tau_\gamma{}^i e_{\alpha\cdot}^{\cdot\gamma} \sqrt{C_1/C} - Q_{i,\xi}^\alpha \tau_\gamma{}^i e_{\alpha\cdot}^{\cdot\gamma} C_1/C +$$
$$\langle P_{i,\xi} g\rangle_{\partial S} \tau^i \sqrt{C_1/C} - a e^{\alpha\beta} \langle \mu f_{\alpha,\beta\xi}\rangle$$

$$R^\alpha = -\frac{\nu}{2} \langle P_i \chi^{\alpha\beta}\rangle_{\partial S} \tau_\beta{}^i - Q^i \tau_{i\beta} H^{\beta\alpha} - \nu \langle P_{i,\xi} (e^\alpha - S_{\beta\cdot}^{\cdot\alpha} g^\beta)\rangle_{\partial S} \tau^i +$$
$$Q_{i,\xi}^\beta \tau^i (\lambda_\beta{}^\alpha - H_{\beta\cdot}^{\cdot\alpha}) + \frac{1}{2} A_{\gamma\cdot}^{\cdot\beta} \langle P_{i,\xi} g^\gamma\rangle_{\partial S} \tau^i J_{\beta\rho}^{(-1)} B^{\rho\alpha}$$

The work done by the forces at the ends $\xi = 0$, $\xi = |\Gamma_0|$ is the same in the improved theory as in the classical theory. In statics the correctness of this step is guaranteed by the Saint-Venant principle. In dynamics, the problem of the boundary conditions requires a special investigation.

4. Effective coefficients of the one-dimensional theory. Below we give the values of the coefficients $J^{\alpha\beta}$ and $H^{\alpha\beta}$ and the corresponding minimizing functions of the variational problem (1.8) in the one-dimensional case for certain transverse cross sections.

1°. *A circle of radius* r

$$J^{\alpha\beta} = \mu \frac{6}{7} |S| \delta^{\alpha\beta}, \quad H^{\alpha\beta} = \frac{\nu}{12} \left[1 + \frac{\nu}{2(1+\nu)} \right] r^2 \delta^{\alpha\beta}$$

$$z = \frac{1}{7} \xi^\alpha \left(1 - \frac{3}{2} \xi^\gamma \xi_\gamma / r^2\right) (2\varphi_\alpha - \nu r^2 \Omega_{\alpha,\xi})$$

2°. *An annulus with radii* r_1 *and* r_2 $(r_1 < r_2)$

$$J^{\alpha\beta} = \mu \frac{6|S|(r_1^2 + r_2^2)^2}{7r_1^4 + 34 r_1^2 r_2^2 + 7 r_2^4} \delta^{\alpha\beta}$$

$$H^{\alpha\beta} = \frac{\nu}{12} \left[\frac{r_1^4 + 10 r_1^2 r_2^2 + r_2^4}{r_2^2 + r_1^2} + \right.$$
$$\left. \frac{\nu}{2(1+\nu)} \frac{7(r_1^8 + r_2^8) + 20 r_1^2 r_2^2 (r_1^4 + r_2^4) - 54 r_1^4 r_2^4}{(r_1^2 + r_2^2)(7r_1^4 + 34 r_1^2 r_2^2 + 7 r_2^4)} \right] \delta^{\alpha\beta}$$

$$z = \left[-1 + \frac{3(r_1^2 + r_2^2)}{7r_1^4 + 34 r_1^2 r_2^2 + 7r_2^4} \left(\frac{3 r_1^2 r_2^2}{\xi^\gamma \xi_\gamma} - \xi^\gamma \xi_\gamma + 3r_1^2 + 3r_2^2 \right) \right] \xi^\alpha \varphi_\alpha +$$
$$\frac{\nu}{3} \left[\frac{1}{2} \xi^\gamma \xi_\gamma - \frac{r_1^4 + r_1^2 r_2^2 + r_2^4}{7r_1^4 + 34 r_1^2 r_2^2 + 7 r_2^4} \left(\frac{3 r_1^2 r_2^2}{\xi^\gamma \xi_\gamma} - \xi^\gamma \xi_\gamma + 3r_1^2 + 3r_2^2 \right) \right] \xi^\alpha \Omega_{\alpha,\xi}$$

3°. *An ellipse* $\xi_1^2 a^2 + \xi_2^2/b^2 \leqslant 1$

$$J_{11} = \mu \frac{3}{2} |S| \frac{3a^2 + b^2}{5a^2 + 2b^2}, \quad H_{11} = \frac{\nu}{6(3a^2 + b^2)} \left[\frac{3}{4} a^4 + \right.$$
$$\left. \frac{3}{2} a^2 b^2 - \left(\frac{1}{4} - \frac{\nu}{1+\nu} \right) b^4 \right]$$

$$z = g_\alpha \varphi^\alpha + e_\alpha \Omega^\alpha_{,\xi}$$

$$g_1 = \xi_1 \frac{4a^2(a^2 + b^2) - 3(3a^2 + b^2)\xi^\gamma \xi_\gamma - (3\xi_2^2 - \xi_1^2)(a^2 - b^2)}{4a^2(5a^2 + 2b^2)}$$

$$e_1 = \xi_1 \frac{\nu}{6} \left[\frac{111 a^4 + 30 a^2 b^2 + 3b^4}{16 a^2 (5a^2 + 2b^2)} \xi^\gamma \xi_\gamma + \right.$$
$$\frac{(31a^4 - 2a^2 b^2 + 3b^4)((a^2 - b^2)(3\xi_2^2 - \xi_1^2) - 12a^2(2a^2 + b^2))}{48 a^2 (3a^2 + b^2)(5a^2 + 2b^2)} +$$
$$\left. \frac{4(a^2 - b^2)(3\xi_2^2 - \xi_1^2 + 3a^2)}{3(3a^2 + b^2)} + \frac{b^2 - a^2}{16} \right]$$

4°. *A rectangle* $|\xi_1| \leqslant a$, $|\xi_2| \leqslant b$

$$J_{11} = \mu \frac{5}{6} |S|, \quad z = g_\alpha \varphi^\alpha + \nu e_\alpha \Omega^\alpha_{,\xi}$$

$$H_{11} = \frac{\nu}{15} a^2 + \frac{3\nu^2}{4(1+\nu)} \Big[\frac{7}{15} a^2 + \frac{2}{9} b^2 + \frac{2b^4}{45a^2} -$$

$$\frac{4b^5}{\pi^5 a^3} \sum_{k=1}^{\infty} \frac{1}{k^5} \operatorname{th} \frac{k\pi a}{b} - \frac{128ab}{\pi^5} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^5} \operatorname{cth} \frac{(2k-1)\pi b}{2a} \Big]$$

$$g_1 = \frac{1}{4} \xi_1 \left(1 - \frac{5}{3} \frac{\xi_1^2}{a^2}\right)$$

$$e_1 = \frac{11}{36} \xi_1^3 - \frac{7}{12} a^2 \xi_1 + \sum_{k=1}^{\infty} (-1)^{k+1} 2b^3 \operatorname{sh}(k\pi\xi_1/b) \times$$

$$\cos(k\pi\xi_2/b) [\pi^3 k^3 \operatorname{ch}(k\pi a/b)]^{-1} + \sum_{k=1}^{\infty} (-1)^{k+1} 16a^2 b \operatorname{ch}((2k-1)\xi_2/2a) \times$$

$$\sin((2k-1)\xi_1/2a) [\pi^3 (2k-1)^3 \operatorname{sh}(2k-1) b/2a]^{-1}$$

The quantities J_{22}, H_{22} and the functions g_2, e_2 for the ellipse and rectangle are obtained by making the subsitution $a \leftrightarrow b$ and the change of indices $1 \leftrightarrow 2$. The remaining components of $J^{\alpha\beta}$, $H^{\alpha\beta}$ are zero with respect to the principal axes of inertia.

5°. *An inhomogeneous rod of rectangular transverse cross section* $|\xi_1| \leqslant a$, $|\xi_2| \leqslant b$ *with shear modulus* μ *depending arbitrarily on the coordinate* ξ_1:

$$J_{11} = \frac{\langle \mu_* \rangle^2}{\langle \mu_*^2 / \mu \rangle}, \quad J_{c2} = \frac{5}{6} \langle \mu \rangle$$

$$z = g_\alpha \varphi^\alpha, \quad g_1 = \frac{\langle \mu_* \rangle}{\langle \mu_*^2 / \mu \rangle} f - \xi_1, \quad g_2 = -\frac{5}{12} \frac{\xi_2^3}{b^2} + \frac{1}{4} \xi_2$$

Here the functions μ_* and f are found from the ordinary differential equations $\mu_{*,1} = \mu\xi_1$, $f_{,1} = \mu_*/\mu$. The integration constants are fixed by the conditions $\mu_*(a) = 0$ and $\langle \mu f \rangle = 0$.

6°. *A rod of rectangular cross section* $|\xi_1| \leqslant a$, $|\xi_2| \leqslant b$, *consisting of three rectangular rods bounded together.* Let the shear modulus μ be a piecewise constant function of ξ_1:

$$\mu = \mu_1, \ -a \leqslant \xi_1 \leqslant -c;$$

$$\mu = \mu_0, \ -c \leqslant \xi_1 \leqslant c; \ \mu = \mu_1, \ c \leqslant \xi_1 \leqslant a$$

$$J_{11} = \mu_0 \frac{5}{6} |S| \varkappa_{11}, \quad J_{22} = \mu_1 \frac{5}{6} |S| \varkappa_{22}$$

$$\varkappa_{22} = 1 + \delta(\alpha - 1), \quad \delta = c/a, \quad \alpha = \mu_0/\mu_1$$

$$\varkappa_{11} = \frac{\alpha[3(1-\delta^5) - 10(1-\delta^3) + 15(1-\delta)] + 8\alpha^2\delta^5 + 20\alpha(1-\delta^2)\delta^3 + 15\delta(1-\delta^2)^2}{8[\alpha + {}^5/_8(\alpha-1)(4\delta^2 - \delta^4 - 3)\delta]^2}$$

Note. The rod discussed in Sect.6 has the following flexural rigidities:

$$\langle E\xi_1\xi_1 \rangle = \frac{8}{3}(1+\nu) a^3 b \mu_1 [1 - \delta^3(1-\alpha)], \quad \langle E\xi_2\xi_2 \rangle = \frac{8}{3}(1+\nu) a b^3 \mu_1 [1 - \delta(1-\alpha)]$$

The flexural and shear energy in the direction of the ξ_1 axis are, respectively, $\langle E\xi_1\xi_1 \rangle \varepsilon^2/h^2$ and $\langle E\xi_1\xi_1 \rangle^2 \varepsilon^2/h^2 l^2 J_{11}$. Their ratio is characterized by the quantity

$$\eta = (l/h)^2 \beta, \quad \beta = \frac{5\alpha\varkappa_{11}}{(1+\nu)[1 - \delta^3(1-\alpha)]} \tag{4.1}$$

If the stress state is such that $\beta \sim (h/l)^2$, then the energy of transverse shear has the same order of smallness as the energy of flexure, and the transverse shear energy must be included in the first approximation. Formula (4.1) shows that β becomes small when the shear moduli have a large gradient. For example, the effect becomes substantial when $\alpha \sim 10^{-2}$, $\beta \sim 10^{-2}$ and for the stress states with $h/l \sim 1/10$.

REFERENCES

1. KIRCHHOFF G., Mechanics. Moscow, Izd-vo Akad. Nauk SSSR, 1962.
2. BERDICHEVSKII V.L., On the energy of an elastic rod. PMM Vol.45, No.4, 1981.
3. TIMOSHENKO S.P., Vibrations in Engineering, Moscow, NAUKA, 1967.
4. DZHANELIDZE G.IU., Kirchhoff relations for naturally twisted rods, and their applications. Tr. Leningr. Politekhn. in-ta, No.1, 1946.
5. BERDICHEVSKII V.L. and STAROSELSKII L.A., On the theory of naturally twisted curvilinear rods. Izv. Akad. Nauk SSSR, MTT, No.6, 1979.
6. BERDICHEVSKII V.L., Variational-asymptotic method of constructing the theory of shells. PMM Vol.43, No.4, 1979.

7. BERDICHEVSKII V.L. and MISYURA V.A., On cross effects between elongation and flexure in the problems of deformation in cylindrical shells. In: Proceedings of the 12-th All Union Conference on the Theory of Shells and Plates. Erevan, 1980. Izd-vo Erevan. un-ta, Erevan, 1980.
8. MISYURA V.A., On the effect of loss of accuracy in the classical theory of shells. Dokl. Akad. Nauk SSSR, Vol.264, No.3, 1982.
9. ELISEEV V.V., Application of the asymptotic method to the problem of the equilibrium of a curvilinear rod. Izv. Akad. Nauk SSSR, MTT, No.3, 1977.
10. BERDICHEVSKII V.L., On the equations of the theory of anisotropic inhomogeneous rods. Dokl. Akad. Nauk SSSR, Vol.228, No.3, 1976.
11. STAROSEL'SKII L.A., On equations describing the oscillations of elastic curvilinear rods. Dokl. Akad. Nauk SSSR, Vol.247, No.1, 1979.
12. BERDICHEVSKII B.L. and KVASHNINA S.S., On equations describing the transverse vibrations of elastic bars. PMM, Vol.40, No.1, 1976.

Translated by L.K.

PMM U.S.S.R., Vol.47, No.6, pp.817-820, 1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 539.374

RIGIDITY IN THE ELASTOPLASTIC TORSION OF SIMPLE RODS*

M.IA. LEONOV and V.D. PEREDERII

Prismatic rods for which the trajectories of tangential stresses under elastic deformation are close to the known trajectories of these stresses in the limiting case of perfect plasticity, are considered. Attention is given to the study of the rigidity of the elastoplastic torsion in the case of perfect plasticity.

The problem of the pure torsion of prismatic inelastic rods occupies a special place among the boundary value problems of mechanics of continuous media, even though it is the simplest of its class; if we exclude the case in which the yield drop is present /1/, then the torsion will not be accompanied by relief of stress; the limiting case of perfectly plastic torsion is statically determinable and can be studied using elementary methods. The appearance of partial plastic deformation formally complicates the problem /2/. However, it is usually the values of the deformation that are of practical interest and not the stresses. It is the deformations that often set a limit to the admissible loads. It is clear that in this context the torsional rigidity is of overriding interset. It can be determined very accurately in an indirect manner, by passing the solution of the partial differential equation at the unknown elastoplastic boundary. The elastic torsion of thin-walled and cylindrical rods when there are no stress concentration foci is investigated in a fairly simple manner in /3/. Plastic deformation reduces the sharpness of the stress concentration and thus widens the range of applicability of the simplified methods of solving elastoplastic problems more efficiently, the higher the level of plastic deformations as compared with elastic deformations. At the centre of the proposed simplification lies the idea of determining the tangential stress trajectories at the periphery of the transverse cross-section in the region of maximum load for elastic as well as the plastic materials; on the contour itself they are identical by virtue of the boundary conditions (the contour is always a trajectory of tangential stresses). The greatest difference between the trajectories under elastic and plastic deformations will occur in the case of perfect plasticity. Nevertheless, the error in determining the torsional rigidity when the actual tangential stress trajectories in the elastic stage are replaced by the trajectories for a perfectly plastic material is practically nil for all singly connected rods with a convex contour, and when parts of the contour are indented with the radius of curvature of the indentations exceeding the distance to the nearest point of the branch of the contour lying opposite /3/. The magnitude of this error represents "the measure of simplicty" of the rod under torsion, and the upper limit of the error when determining the torsional rigidity of the inealstic rods. The more plastic the material (i.e. the greater the plastic deformations), the smaller the error in determining the torsional rigidity; in the limiting case of infinitely large deformations without reinforcement it tends to zero. The present paper deals with the case of linear reinforcement, but the computations are carried out for perfect plasticity.

**Prikl.Matem.Mekhan.*, 47, 6, 1025-1029, 1983